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INTERACTION OF NON-LINEAR WAVES AND THE FACTORIZATION METHOD†

A. A. LOKSHIN

Moscow

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This note consists of two parts. In the first part we consider the interaction of incident and reflected waves in a non-linear rod. Exact formulae are obtained (and also estimates) for the extremal values of the stress in the region in which the incident and reflected ways interact. In the second part, which is a continuation of [1–6], a special asymptotic factorization of the non-linear wave equation is constructed, which enables an ordinary second-order differential equation to be derived, which describes the interaction of a short pulse and a simple wave moving in opposite directions. The amplitudes of the waves are assumed to be finite.

The method used in the first part of this paper is a development of an idea from [7], where the problem of calculating the maximum level of water on a fixed wall when a simple wave rolls against this wall is considered. (See also [8, p. 34], where the results of [7] are presented.) Note, however, that the method used in [7] is only suitable for calculating the maximum amplitude of the resultant wave at the limit of the region (on the wall of the channel). However, the method proposed in the present paper enables one to investigate the extremal values over the whole region in which the incident and reflected waves interact.

1. Consider a non-linearly elastic rod of density ρ = const, situated on the segment $0 \leqslant x \leqslant L$ of the x-axis; we will take the constitutive equation for the material of the rod in the form $\varepsilon = a(\sigma)$. Here ε is the strain and σ is the stress; the non-linear function $a(\sigma)$ satisfies the conditions $a(0) = 0$, $0 < a'(\sigma) \leqslant$ const.

The equations of motion of the rod have the following form in Lagrangian coordinates

$$\rho\frac{\partial \upsilon}{\partial t} = \frac{\partial \sigma}{\partial x}, \quad \frac{\partial \upsilon}{\partial x} = \frac{\partial a(\sigma)}{\partial t} \tag{1.1}$$

where υ is the velocity of a material element. We specify the initial conditions in the form

$$\upsilon(0, x) = 0, \quad \sigma(0, x) = 0 \tag{1.2}$$

Finally, we write the boundary conditions for the stresses

$$\sigma(t, 0) = f(t), \quad \sigma(t, L) = 0 \tag{1.3}$$

Here $f(t)$ is a smooth function which vanishes when $t \leqslant 0$ and when $t \geqslant T > 0$ (we assume that the function $f(t)$ differs from identical zero in the segment $[0, T]$).

It is required to obtain (or estimate) the extremal values of the stress σ_{max} and σ_{min} on the set D—the closure of the region of interaction of the incident and reflected wave (see Fig. 1). It is assumed here that the wave reflected from the end of the rod $x = L$ first reaches the end of the rod $x = 0$ when $t > T$. Moreover, it is assumed that no shockwaves occur in D.

We will introduce the following notation

$$\psi(\sigma) = 2\int_0^\sigma \sqrt{\frac{a'(\sigma)}{\rho}}\, d\sigma$$
$$f_{max} = \max_{0 \leqslant t \leqslant T} f(t); \quad f_{min} = \min_{0 \leqslant t \leqslant T} f(t) \tag{1.4}$$

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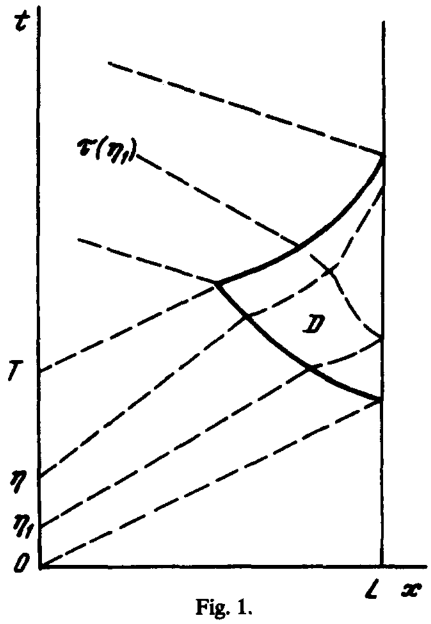

Fig. 1.

Suppose t^* is the least value of $t \in [0, T]$ for which $f(t) = f_{\min}$, while t^{**} is the least value of $t \in [0, T]$ for which $f(t) = f_{\max}$. Since $f(t) \neq 0$ $[0, T]$ we have $t^* \neq t^{**}$.

Theorem 1. Suppose the above assumptions hold. Then, if $t^* < t^{**}$ we have

$$\sigma_{\max} = \psi^{-1}(\psi(f_{\max}) - \psi(f_{\min})), \quad \sigma_{\min} \geqslant \psi^{-1}(\psi(f_{\min}) - \psi(f_{\max})) \tag{1.5}$$

If $t^* > t^{**}$, then in the first of relations (1.5) the equality sign must be replaced by $\leqslant$, while in the second the $\geqslant$ sign must be replaced by the equality sign.

Proof. We will introduce Riemann invariants for system (1.1) by the formulae

$$r = \psi(\sigma)/2 + \upsilon, \quad s = \psi(\sigma)/2 - \upsilon \tag{1.6}$$

As we know, the invariants r and s are constant on the corresponding families of characteristics, namely

$$\begin{aligned} &r = r(\tau) \\ &\tau(t, x) = \text{const} \ \text{ when } \ dt\,/\,dx = -\sqrt{\rho a'(\sigma)}; \quad \tau = t \ \text{ when } \ x = L \end{aligned} \tag{1.7}$$

Similarly

$$\begin{aligned} &s = s(\eta) \\ &\eta(t, x) = \text{const} \ \text{ when } \ dt\,/\,dx = \sqrt{\rho a'(\sigma)}; \quad \eta = t \ \text{ when } \ x = 0 \end{aligned} \tag{1.8}$$

It follows from (1.6)–(1.8) that

$$\sigma(t, x) = \psi^{-1}(r(\tau) + s(\eta)) \tag{1.9}$$

We will now consider the characteristic of positive slope emerging from the point $t = \eta, x = 0$, where $0 \leqslant \eta \leqslant T$ (see Fig. 1). Suppose this characteristic intersects the line $x = L$ when $t = \tau$. Then, the quantity τ in D, which is retained on characteristics of negative slope, turns out to be a function of $\eta : \tau = \tau(\eta)$ when $(t, x) \in D$. We will take an arbitrary point $(t, x) \in D$ and draw through it characteristics of positive and negative slope. Suppose the characteristic of positive slope intersects the line $x = 0$ when $t = \eta$, while the characteristic of negative slope intersects the line $x = L$ when $t = \tau(\eta_i)$. Hence, (1.9) takes the form

$$\sigma(t, x) = \psi^{-1}\{r(\eta_1)) + s(\eta)\} \tag{1.10}$$

It is geometrically obvious from Fig. 1 that $0 \leqslant \eta_1 \leqslant \eta$ and that

$$\eta_1 = \eta \quad \text{when} \quad x = L \tag{1.11}$$

We will now express $s(\eta)$ in terms of the boundary value on the left end of the rod (the first of conditions (1.3)). In a simple wave, emerging from the end $x = 0$, we obviously have $r = 0$, whence, by virtue of (1.6), we have $\upsilon = -\psi(\sigma)/2$. Consequently, in the simple wave mentioned $s = \psi(\sigma)$. Assuming here that $x = 0$ and taking into account the fact that $\eta = t$ when $x = 0$, by virtue of the first of conditions (1.3) we have

$$s(\eta) = \psi(f(\eta)), \quad 0 \leqslant \eta \leqslant T \tag{1.12}$$

We now use the boundary condition on the right (free) end of the rod. From the second of conditions (1.3) and (1.10), (1.11) we have

$$r(\tau(\eta)) + s(\eta) = 0$$

whence, in view of (1.12), we obtain

$$r(\tau(\eta)) = -\psi(f(\eta)), \quad 0 \leqslant \eta \leqslant T \tag{1.13}$$

Substituting (1.12) and (1.13) into (1.10) we have in D

$$\sigma(t, x) = \psi^{-1}\{\psi(f(\eta)) - \psi(f(\eta_1))\} \tag{1.14}$$

Suppose now that $t^* < t^{**}$. It is then obvious that the characteristic of positive slope, emerging from the point $t = t^{**}, x = 0$, intersects in D the characteristic of negative slope emerging from the point $t = t^*, x = L$. Now, by virtue of the fact that the functions ψ^{-1} and ψ are monotonic (which follows from the fact that $a(\sigma)$ is monotonic), we have that the maximum value of $\sigma(t, x)$ in D is reached precisely at the point where the above two characteristics intersect. We have thereby proved the first of relations (1.5). The correctness of the second of relations (1.5) follows directly from (1.14). The corresponding relations for the case when $t^* > t^{**}$ can be proved similarly. This proves the theorem.

Note. We can investigate problems of the interaction of incident and reflected waves for other boundary conditions in exactly the same way. For example, instead of the problem for a rod with a free end $x = L$, we could have considered the problem when one end was clamped, the problem when a point load is applied at the end of the rod, etc.

2. Consider the equation

$$\frac{\partial^2 a(\sigma)}{\partial t^2} - \frac{1}{\rho}\frac{\partial^2 \sigma}{\partial x^2} = 0 \tag{2.1}$$

which is a consequence of system (1.1). The possibility of exact factorization of Eq. (2.1) was established in [1], and a special asymptotic factorization of this equation was investigated in [5]. Here we will establish the following result, which is more general than that in [5].

Theorem 2. In the region in which $\sigma = \sigma(t, x)$ varies smoothly, the following identity holds

$$\frac{\partial^2 a(\sigma)}{\partial t^2} - \frac{1}{\rho}\frac{\partial^2 \sigma}{\partial x^2} + N(t,x,\sigma) \equiv \left\{\frac{\partial}{\partial t}\sqrt{a'(\sigma)} \mp \frac{1}{\sqrt{\rho}}\frac{\partial}{dx} + C(t,x)\frac{d}{d\sigma}[a'(\sigma)]^{-1/4}\right\}\times$$

$$\times\left\{\sqrt{a'(\sigma)}\frac{\partial}{\partial t} \pm \frac{1}{\sqrt{\rho}}\frac{\partial\sigma}{dx} + C(t,x)[a'(\sigma)]^{-1/4}\right\} \tag{2.2}$$

$$N(t,x,\sigma) \equiv -C^2(t,x)\frac{a''(\sigma)}{4[a'(\sigma)]^{3/2}} + [a'(\sigma)]^{-1/4}C_t'(t,x) \mp \frac{1}{\sqrt{\rho}}[a'(\sigma)]^{-1/4}C_x'(t,x) \tag{2.3}$$

where $C(t, x)$ is an arbitrary smooth function. (Either the upper or lower signs are chosen simultaneously in (2.2) and (2.3).)

Notes. 1. In this paper we make the following agreement: in the product of operators (of the differentiation or multiplication by a function) the operator on the right acts earlier. For example

$$\frac{\partial}{\partial t}\sqrt{a'(\sigma)}\sqrt{a'(\sigma)}\frac{\partial\sigma}{\partial t} \equiv \frac{\partial^2 a(\sigma)}{\partial t^2}$$

2. It follows from (2.3) that in the special case when $C(t, x) \equiv 0$, we have $N \equiv 0$, and the representation (2.2) becomes an exact factorization of the non-linear wave operator established in [1]. When $C(t, x) = [a'(\rho g(L-x))]^{1/4}$, g = const, representation (2.2) reduces to the result obtained in [5].

3. We will henceforth adhere to the following agreement, regarding the notation of derivatives. If $f = f(t, x, \sigma)$, then f_t' and f_x' denote partial derivatives with σ = const, whereas

$$\frac{\partial f}{\partial t} = f_t' + f_\sigma' \frac{\partial \sigma}{\partial t}, \quad \frac{\partial f}{\partial x} = f_x' + f_\sigma' \frac{\partial \sigma}{\partial x}$$

Proof of the theorem. We will present some considerations which we will use to obtain the identity (2.2), (2.3), confining ourselves, for simplicity, to the case of the upper signs. Consider the products

$$\left\{\frac{\partial}{\partial t}\sqrt{a'(\sigma)} - \frac{1}{\sqrt{\rho}}\frac{\partial}{\partial x} + \varphi\right\}\left\{\sqrt{a'(\sigma)}\frac{\partial \sigma}{\partial t} + \frac{1}{\sqrt{\rho}}\frac{\partial \sigma}{\partial x} + \psi\right\} \tag{2.4}$$

where $\varphi = \varphi(t, x, \sigma)$, $\psi = \psi(t, x, \sigma)$ and we will endeavour to choose the previously undefined functions φ and ψ so that expression (2.4) differs from the non-linear wave operator from (2.1) only in terms not containing derivatives of $\sigma(t, x)$. We multiply the operator brackets in (2.4), adhering to the above-mentioned agreements. The product (2.4) then takes the form

$$\begin{gathered}\frac{\partial^2 a(\sigma)}{\partial t^2} - \frac{1}{\sqrt{\rho}}\frac{\partial}{\partial x}\sqrt{a'(\sigma)}\frac{\partial \sigma}{\partial x} + \varphi\sqrt{a'(\sigma)}\frac{\partial \sigma}{\partial t} + \frac{\partial}{\partial t}\sqrt{a'(\sigma)}\frac{1}{\sqrt{\rho}}\frac{\partial \sigma}{\partial x} - \\ -\frac{1}{\rho}\frac{\partial^2 \sigma}{\partial x^2} + \varphi\frac{1}{\sqrt{\rho}}\frac{\partial \sigma}{\partial x} + \frac{\partial}{\partial t}\sqrt{a'(\sigma)}\psi - \frac{1}{\sqrt{\rho}}\frac{\partial \psi}{\partial x} + \varphi\psi\end{gathered} \tag{2.5}$$

Note that, since the following general identity holds [1]

$$\frac{\partial}{\partial t}f(\sigma)\frac{\partial \sigma}{\partial x} \equiv \frac{\partial}{\partial x}f(\sigma)\frac{\partial \sigma}{\partial t}$$

the second and fourth terms in (2.5) cancel one another out. Further, the sum of the third and seventh terms in (2.5) can obviously be rewritten in the form

$$\left\{\varphi\sqrt{a'} + (\sqrt{a'}\psi)_\sigma'\right\}\frac{\partial \sigma}{\partial t} + \sqrt{a'}\psi_t' \tag{2.6}$$

while the sum of the sixth and eighth terms in (2.5) can be written in the form

$$\left\{\varphi\frac{1}{\sqrt{\rho}} - \psi_\sigma'\frac{1}{\sqrt{\rho}}\right\}\frac{\partial \sigma}{\partial x} - \frac{1}{\sqrt{\rho}}\psi_x' \tag{2.7}$$

We equate the contents of the braces in (2.6) and (2.7) to zero

$$\varphi\sqrt{a' + (\sqrt{a'}}\psi')_\sigma' = 0, \quad \varphi = \psi_\sigma' \tag{2.8}$$

It is clear that if φ and ψ satisfy Eqs (2.8), expression (2.5) will differ from the left-hand side of (2.1) by the quantity

$$N = \varphi\psi + \sqrt{a'(\sigma)}\psi_t' - \psi_x' / \sqrt{\rho} \tag{2.9}$$

Hence, it only remains to solve system (2.8). Note that Eqs (2.8) are ordinary differential equations (in σ), while the variables t and x only occur in them as parameters. Substituting the expression for φ from the second equation of (2.8) into the first, we obtain an equation with separated variables. Integrating it we obtain

$$\psi = C(t, x)[a'(\sigma)]^{-1/4} \tag{2.10}$$

whence, by virtue of the second equation of (2.8), we have

$$\varphi = C(t,x)\frac{d}{d\sigma}[a'(\sigma)]^{-1/4} \tag{2.11}$$

Finally, substituting the functions (2.10) and (2.11) into (2.9) we obtain the required expression for the discrepancy N. This proves the theorem.

We now formulate the following boundary conditions for Eq. (2.1)

$$\sigma(t,0) = \sigma_0(t/\gamma), \quad 0<\gamma \ll 1 \tag{2.12}$$

$$\sigma(t,L) = \sigma_1(t) \tag{2.13}$$

Here $\sigma_0(t)$ is a smooth function which vanishes when $t \leqslant 0$ and when $t \geqslant T > 0$; $\sigma_1(t)$ is a smooth function which vanishes when $t \leqslant 0$; finally $L = \text{const} > 0$. Moreover, we write the initial conditions

$$\sigma(0,x) = \sigma'_t(0,x) = 0 \tag{2.14}$$

When investigating this problem we will confine ourselves to times during which no shock waves occur and, moreover, impulses, specified on each of the boundaries, cannot be reflected from the opposite boundary.

We will further denote the solution of problem (2.1), (2.13), (2.14), defined on the ray $x \leqslant L$, by $\bar\sigma(t,x)$. It is obvious that before discontinuities of the function $\bar\sigma(t,x)$ occur the equation of a simple wave, propagating to the left

$$\sqrt{a'(\bar\sigma)}\frac{\partial\bar\sigma}{\partial t} - \frac{1}{\sqrt{\rho}}\frac{\partial\bar\sigma}{\partial x} = 0 \tag{2.15}$$

is satisfied, and can be solved analytically by the method of characteristics.

We now consider the expression in the inner braces in the product of the right-hand side of (2.2) when the upper signs are chosen and we equate it to zero. It is obvious that the equation obtained

$$\sqrt{a'(\sigma)}\frac{\partial\sigma}{\partial t} + \frac{1}{\sqrt{\rho}}\frac{\partial\sigma}{\partial x} + \frac{C(t,x)}{[a'(\sigma)]^{1/4}} = 0 \tag{2.16}$$

describes a certain wave propagating to the right. We will now require that the function $\bar\sigma(t,x)$, introduced above, should satisfy Eq. (2.16). Since the function $C(t,x)$ has so far remained arbitrary, we can obviously put

$$C(t,x) = -\left[\sqrt{a'(\bar\sigma)}\frac{\partial\bar\sigma}{\partial t} + \frac{1}{\sqrt{\rho}}\frac{\partial\bar\sigma}{\partial x}\right][a'(\bar\sigma)]^{1/4} \tag{2.17}$$

whence, in view of (2.15), we have

$$C(t,x) = -\frac{2}{\sqrt{\rho}}\frac{\partial\bar\sigma}{\partial x}[a'(\bar\sigma)]^{1/4}$$

Further, we note that for the function $C(t,x)$, defined by (2.17), we have

$$N(t,x,\bar\sigma) \equiv 0 \tag{2.18}$$

In fact, on the one hand, the function $\bar\sigma$ makes the non-linear wave operator (2.1) vanish, while on the other hand, it is the solution of Eq. (2.16). Hence, the assertion follows from identity (2.2).

It is now clear that the problem of the interaction of the waves (2.1) and (2.12)–(2.14) reduces asymptotically to the much more simple problem (2.12) and (2.16) (where the function $C(t,x)$ is defined by (2.17)).

In fact, let us denote the solution of problem (2.12), (2.16) by $\tilde\sigma(t,x)$ and substitute $\tilde\sigma(t,x)$ into identity (2.2), where we have chosen the upper signs. It is clear that the right-hand side of identity (2.2) vanishes when $\sigma = \tilde\sigma(t,x)$ is substituted. It therefore follows from (2.2) that

$$\frac{\partial^2 a(\tilde\sigma)}{\partial t^2} - \frac{1}{\rho}\frac{\partial^2\tilde\sigma}{\partial x^2} + N(t,x,\tilde\sigma) = 0 \tag{2.19}$$

It is further obvious that, outside a narrow strip, in which the support of the short impulse, moving to the right, is concentrated, we have $\tilde{\sigma}(t,x) = \bar{\sigma}(t,x)$. Hence, by virtue of (2.18), it follows from (2.19) that $\tilde{\sigma}(t,x)$ satisfies Eq. (2.1) outside the narrow strip indicated.

On the other hand, in the narrow strip where the support of the short pulse considered is situated, terms of the non-linear wave operator $\partial_t^2 a(\tilde{\sigma})$ and $\rho^{-1}\partial_x^2\tilde{\sigma}$ obviously are of the order of $1/\gamma^2$, whereas $N(t,x,\tilde{\sigma}) = O(1)$ (since, as was noted above, derivatives of the function $\tilde{\sigma}$ do not occur in the expression for $N(t,x,\tilde{\sigma})$ Consequently, we can neglect the discrepancy N in (2.19), whence it follows that $\tilde{\sigma}$ is the asymptotic solution of Eq. (2.1).

Finally, the satisfaction of both boundary conditions (2.12) and (2.13) and both initial conditions (2.14) for the function $\tilde{\sigma}(t,x)$ follows from its construction.

The solution $\tilde{\sigma}$ can obviously be constructed by the method of characteristics, namely, the equations of the characteristics are such that

$$\frac{dt}{dx} = \sqrt{\rho a'(\sigma)}, \quad \frac{d\tilde{\sigma}}{dx} = -\frac{\sqrt{\rho}}{[a'(\sigma)]^{1/4}} C(t,x) \tag{2.20}$$

Here $\tilde{\sigma}(t,0) = \sigma_0(t/\gamma)$ while the function $C(t,x)$, as before, is given by (2.17).

Note. It is obvious that system (2.20) reduces to a single second-order equation

$$\frac{d}{dx}F\left(\frac{1}{\sqrt{\rho}}\frac{dt}{dx}\right) = -\sqrt{\rho}\,C(t,x)$$

$$\left(F = g\cdot f,\ g(\sigma) = \int_0^{\sigma}[a'(\sigma)]^{1/4}d\sigma,\ f^{-1}(\sigma) = \sqrt{a'(\sigma)}\right)$$

In conclusion, we will consider the special case when the amplitude of the pulse σ_1 is so small that (outside the region of interaction with the pulse σ_0) this pulse propagates without non-linear distortions. In other words, suppose

$$a'(\sigma) = \frac{1}{E} = \text{const} \quad \text{for} \quad |\sigma| \leq \max_t |\sigma_1(t)| \tag{2.21}$$

Then, obviously, at times preceding the reflection of the pulse σ_1 from the boundary $x = 0$

$$\bar{\sigma} = \sigma_1(t + (x - L)\sqrt{\rho/E})$$

However, substituting the last equality into (2.17) and assuming that

$$y = t + x\sqrt{\rho/E} \tag{2.22}$$

we can rewrite the equations of the characteristics (2.20) in the form

$$\begin{aligned} \frac{dy}{dx} &= \sqrt{\rho}\left(\frac{1}{\sqrt{E}} + \sqrt{a'(\tilde{\sigma})}\right) \\ \frac{d\tilde{\sigma}}{dx} &= \frac{2\sqrt{\rho}}{[a'(\tilde{\sigma})]^{1/4}E^{3/4}}\sigma_1'(y - L\sqrt{\rho/E}) \end{aligned} \tag{2.23}$$

These equations (unlike the more general equations (2.20)) can be integrated in quadratures. In fact, dividing the second of equations (2.23) by the first and integrating, we have

$$\frac{E^{3/4}}{2}\int_{\sigma_0(\tau/\gamma)}^{\tilde{\sigma}}[a'(\sigma)]^{1/4}\left(\frac{1}{\sqrt{E}} + a'(\sigma)\right)d\sigma = \sigma_1\left(y - \frac{L}{\sqrt{E/\rho}}\right) - \sigma_1\left(\tau - \frac{L}{\sqrt{E/\rho}}\right) \tag{2.24}$$

Here we have denoted the value of t (and hence y) at $x = 0$ by τ. Relation (2.24) obviously defines the function $\tilde{\sigma} = b(\tau, y)$. Substituting this expression for $\tilde{\sigma}$ into the first of equations (2.23) and integrating, we obtain

$$\frac{1}{\sqrt{\rho}}\int_{\tau}^{y}\frac{dy}{1/\sqrt{E} + \sqrt{a'(b(\tau,y))}} = x \tag{2.25}$$

Formulae (2.22), (2.24) and (2.25) also define the required asymptotic solution of the problem formulated above.

It can be verified that, in the purely linear case (i.e. when $a'(\sigma) = 1/E = \text{const}$) at times preceding the reflection of pulses from the boundaries $x = 0$ and $x = L$, relations (2.22), (2.24) and (2.25) give

$$\tilde{\sigma} = \sigma_0\left(\frac{t - x\sqrt{\rho / E}}{\gamma}\right) + \sigma_1(t + x\sqrt{\rho / E})$$

i.e. the asymptotic solution constructed is accurate.

Note that the proposed method can also be extended to the case of non-linear wave equations with variable coefficients.

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